Math 2E Quiz 9 Afternoon - May 26th

Please write your name and ID on the front.

Show all of your work, and simplify all your answers. *There is a question on the back side.

1. Recall that $\nabla \bullet (\nabla \times \mathbf{F}) = 0$, this is always zero (when the components of $\mathbf{F}$ have continuous 2nd order partial derivatives). Consider $\mathbf{F}(x, yz) = < xyz^4,\ x^2z^4,\ 4x^2yz^3 >$.

(a) [4pts] Compute $\nabla \times \mathbf{F}$. Is $\mathbf{F}$ conservative?

$$\nabla \times \vec{F} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ xyz^4 & x^2z^4 & 4x^2yz^3 \end{bmatrix}$$

$$= (4x^2z^3 - 4x^2z^3)\hat{i} + (4xyz^3 - 8xyz^3)\hat{j} + (2xz^4 - xz^4)\hat{k}$$

+1 +1 +1

$$= \boxed{0\hat{i} - 4xyz^3\hat{j} + xz^4\hat{k}}; \quad \nabla \times \vec{F} \neq (0,0,0), \ \vec{F} \text{ is not conservative.}$$

+1

(b) [2pts] Verify that $\nabla \bullet (\nabla \times \mathbf{F}) = 0$ by taking the divergence of your answer in (a).

$$\nabla \cdot (\nabla \times \vec{F}) = \nabla \cdot < 0, -4xyz^3, xz^4 >$$ +1

$$= 0 - 4xz^3 + 4xz^3 = 0$$ +1 ✓

(c) [4pts] Prove that $\nabla \bullet (f\mathbf{G}) = \nabla f \bullet \mathbf{G} + f\ \nabla \bullet \mathbf{G}$.
Here, f is a scalar function on $\mathbb{R}^3$ and $\mathbf{G} = < P, Q, R >$ is a vector field on $\mathbb{R}^3$.
(Assume that f and the components P, Q, R are smooth functions).

Pf: $\nabla \cdot (f\vec{G}) = \nabla \cdot < fP, fQ, fR >$ +1

$$= \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR)$$

$$= f_xP + fP_x + f_yQ + fQ_y + f_zR + fR_z$$ +1

$$= (f_xP + f_yQ + f_zR) + f(P_x + Q_y + R_z)$$ +1

+1

$$= (\nabla f \cdot \vec{G}) + f(\nabla \cdot \vec{G})$$ ✓

(Note: This S is also the same as $z = xy$ if we eliminate the parameter.)

2. Consider the surface S parameterized by $\mathbf{r}(u, w) = < u,\ w,\ uw >$, $u^2 + w^2 \leq 1$.

(a) [3pts] Compute $\mathbf{r_u} \times \mathbf{r_w}$.

+1 $\vec{r}_u = <1, 0, w>$

+1 $\vec{r}_w = <0, 1, u>$

$$\Rightarrow \vec{r}_u \times \vec{r}_w = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & w \\ 0 & 1 & u \end{bmatrix}$$

$$= \boxed{-w\hat{i} - u\hat{j} + \hat{k}}$$

+1

(b) [3pts] Find the equation of the tangent plane at the point $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$.

At $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, $\boxed{u = \frac{1}{2},\ w = \frac{1}{3}}$ +1 $\Rightarrow \vec{r}_u \times \vec{r}_w \Big|_{\substack{\text{At} \\ u = 1/2 \\ w = 1/3}} = < -\frac{1}{3}, -\frac{1}{2}, 1 >$

$(uw = 1/6 \checkmark)$

Normal Vector!

+1

Thus, the plane equation is $\boxed{-\frac{1}{3}(x - \frac{1}{2}) - \frac{1}{2}(y - \frac{1}{3}) + 1(z - \frac{1}{6}) = 0}$

(c) [4pts] Compute the surface area of S on the given domain $u^2 + w^2 \leq 1$.

Def: $\text{Area}(S) = \iint_{D(u,w)} |\vec{r}_u \times \vec{r}_w| \, dA$. From part (a), we see:

$|\vec{r}_u \times \vec{r}_w| = \sqrt{w^2 + u^2 + 1}$. +1 Note: this and the domain $u^2 + w^2 \leq 1$ imply polar.

Thus, $\text{Area}(S) = \iint_{\substack{u^2 + w^2 \leq 1 \\ \text{domain}}} \sqrt{u^2 + w^2 + 1} \, dA$ +1

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \sqrt{1 + r^2} \, r \, dr \, d\theta \quad +1$$

$u = 1 + r^2$

$du = 2r\,dr$

$$= \int_0^{2\pi} \int_{u=1}^{2} \sqrt{u} \cdot \frac{du}{2} \, d\theta = \int_{\theta=0}^{2\pi} \frac{\cancel{2} u^{3/2}}{3} \cdot \frac{1}{\cancel{2}} \Bigg|_{u=1}^{2} d\theta$$

θ-indep.

$$= \boxed{2\pi \left[\frac{2\sqrt{2} - 1}{3} \right] \text{ units}^2} \quad +1$$